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## Arithmetic and moduli of elliptic surfaces

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## CHAPTER 3

### Kuwata's surfaces

#### 1. Introduction

In this chapter we assume that  $K$  is a field of characteristic 0.

We introduce elliptic surfaces  $\pi_n : X_n \rightarrow \mathbf{P}^1$  defined over  $K$ , for every integer  $n \geq 1$ . In the special case  $n \equiv 0 \pmod{2}$ , the function field extension  $K(X_n)/K(\mathbf{P}^1)$  is  $K(x, z, t)/K(t)$  where  $x^3 + ax + b - t^n(z^3 + cz + d) = 0$ , for some  $a, b, c, d \in K$ . For every  $n \geq 1$  the surface  $X_n$  is birational to a base-change  $X_1 \times_{\mathbf{P}^1} \mathbf{P}^1$ , where  $\mathbf{P}^1 \rightarrow \mathbf{P}^1$  is an  $n$ -cyclic cover.

Kuwata [41] computed the rank of the Mordell-Weil groups  $MW(\pi_i) := MW_{\overline{K}}(\pi_i)$  of  $\pi_i$ , for  $i = 1, \dots, 6$ , and he gave a strategy for computing a rank 12 subgroup of the rank 16 group  $MW(\pi_6)$ . This strategy for finding generators of  $MW(\pi_6)$  is already mentioned in an email correspondence between Jasper Scholten and Masato Kuwata. We describe and extend their ideas in order to describe generators of the Mordell-Weil groups  $MW(\pi_i)$  for  $i = 2, 3, 4$  and 6. Most of the computations we use, can be found in the Maple worksheet [32]. We indicate how one can find in special cases generators for the Mordell-Weil group of  $\pi_5$ . Finally, in the case  $K = \mathbf{Q}$  we discuss how large the part of the Mordell-Weil group consisting of  $\mathbf{Q}$ -rational sections can be.

For  $i \leq 6$  the  $\pi_i : X_i \rightarrow \mathbf{P}^1$  are elliptic  $K3$  surfaces. There are very few examples of such surfaces where generators for the Mordell-Weil group have been found.

#### 2. Notation and results

Fix two elliptic curves  $E$  and  $F$ , with  $j(E) \neq j(F)$ . Let  $\iota : E \times F \rightarrow E \times F$  be the automorphism sending  $(P, Q) \rightarrow (-P, -Q)$ . The minimal desingularization of  $(E \times F)/\langle \iota \rangle$  is a  $K3$  surface which we denote by  $Y$ . It is called the Kummer surface of  $E \times F$ .

The surface  $Y$  possesses several elliptic fibrations. For a generic choice of  $E$  and  $F$ , all fibrations on  $Y$  are classified by Oguiso ([52]). Following Kuwata [41], we concentrate on a particular fibration  $\psi : Y \rightarrow \mathbf{P}^1$  having two fibers of type  $IV^*$  and 8 other irreducible singular fibers. Assume that the fibers of type  $IV^*$  are over 0 and  $\infty$ . Let  $\pi_6 : X_6 \rightarrow \mathbf{P}^1$  be the cyclic degree 3 base-change of  $\psi$  ramified over 0 and  $\infty$ . Assume that  $E$  is given by  $y^2 = x^3 + ax + b$  and  $F$  is given by  $y^2 = x^3 + cx + d$ . Set  $B(t) = \Delta(F)t + 864bd + \Delta(E)/t$ , with  $\Delta(E)$  the discriminant of  $E$ , i.e.,  $\Delta(E) = -16(4a^3 + 27b^2)$ , and  $\Delta(F) = -16(4c^3 + 27d^2)$ . Then a Weierstrass equation for  $\pi_6$  is

$$y^2 = x^3 - 48acx + B(t^6).$$

Define  $\pi_i : X_i \rightarrow \mathbf{P}^1$  as the elliptic surface associated to the Weierstrass equation

$$y^2 = x^3 - 48acx + B(t^i).$$

Clearly, interchanging the role of  $E$  and  $F$  corresponds to the automorphism  $t \mapsto 1/t$  on  $X_i$ . One has that  $j(\pi_2) = j(\psi)$ , but  $X_2$  and  $Y$  are *not* isomorphic as fibered surfaces.

By construction, it is clear that  $MW(\pi_n)$  can be regarded as a subgroup of  $MW(\pi_{nm})$  for  $m \geq 1$ .

We recall the following result from [41], most of which will be reproven in the course of Section 4.

**THEOREM 2.1** (Kuwata, [41, Theorem 4.1]). *The surface  $X_i$  is a K3 surface if and only if  $i \leq 6$ . Set*

$$h = \begin{cases} 0 & \text{if } E \text{ and } F \text{ are not isogenous,} \\ 1 & \text{if } E \text{ and } F \text{ are isogenous and } E \text{ does not admit complex multiplication,} \\ 2 & \text{if } E \text{ and } F \text{ are isogenous and } E \text{ admits complex multiplication.} \end{cases}$$

Suppose  $j(E) \neq j(F)$ , then

$$\text{rank } MW(\pi_i) = \begin{cases} h & \text{if } i = 1, \\ 4 + h & \text{if } i = 2, \\ 8 + h & \text{if } i = 3, \\ 12 + h & \text{if } i = 4, \\ 16 + h & \text{if } i = 5, \\ 16 + h & \text{if } i = 6. \end{cases}$$

In the case that  $j(E) = j(F)$  then the ranks of  $MW(\pi_i)$  tend to be lower (see [41, Theorem 4.1]). Kuwata in [41] restricts himself to the  $X_i$  with  $i \leq 6$ . We observe the following corollary of his results.

**COROLLARY 2.2.** *The rank of  $MW(\pi_{60})$  is at least  $40 + h$ .*

**PROOF.** It can be easily seen that the rank  $40 + h$  group

$$MW(\pi_1) \oplus \frac{MW(\pi_2)}{MW(\pi_1)} \oplus \frac{MW(\pi_4)}{MW(\pi_2)} \oplus \frac{MW(\pi_5)}{MW(\pi_1)} \oplus \frac{MW(\pi_6)}{MW(\pi_2)}.$$

injects into  $MW(\pi_{60})$ . (The summands correspond to different eigenspaces for the induced action of  $t \mapsto \zeta_{60}t$  on  $MW(\pi_{60})$ .)  $\square$

**REMARK 2.3.** The highest known Mordell-Weil rank for an elliptic surface  $\pi : X \rightarrow \mathbf{P}^1$ , such that  $j(\pi)$  is non-constant is 56, a result due to Stiller [78]. The geometric genus of our example is very low compared to for example Stiller's examples: one can easily show that  $p_g(X_{60}) = 19$ , while Stiller's examples have  $p_g + 1$  divisible by 210.

Our aim is to provide explicit equations for generators of  $MW(\pi_i)$ , for several small values of  $i$ , in terms of the parameters  $a, b, c, d$  of the two elliptic curves

$$E : y^2 = x^3 + ax + b \text{ and } F : y^2 = x^3 + cx + d.$$

The main idea used in the sequel is the following. Let  $\pi : S \rightarrow \mathbf{P}^1$  be a K3 surface. Let  $\sigma$  be an automorphism of finite order of  $S$ , such that we have a commutative diagram

$$\begin{array}{ccc} S & \xrightarrow{\sigma} & S \\ \downarrow \pi & & \downarrow \pi \\ \mathbf{P}^1 & \xrightarrow{\tau} & \mathbf{P}^1 \end{array}$$

and the order of  $\tau$  equals the order of  $\sigma$ . We obtain an elliptic surface  $\psi : S' := \widetilde{X/\langle \sigma \rangle} \rightarrow \mathbf{P}^1/\langle \tau \rangle \cong \mathbf{P}^1$ . In the case that  $S'$  is a rational surface, it is in principle possible to find

explicit equations for generators of  $MW(\psi)$ . One can pull back sections of  $\psi$  to sections of  $\pi$ , which establishes  $MW(\psi)$  as a subgroup of  $MW(\pi)$ . In the case that  $\sigma$  is of even order then there exists a second elliptic surface,  $\psi'' : S'' \rightarrow \mathbf{P}^1$ , such that  $MW(\psi) \oplus MW(\psi'')$  modulo torsion injects into  $MW(\pi)$  modulo torsion. (This process is called twisting and is discussed in more detail in Section 1.5.)

In [41, Section 5] it is indicated how one can find generators for a rank 12 subgroup of  $MW(\pi_6)$  by using the above strategy for the involutions  $(x, y, t) \mapsto (x, y, -t)$ , and  $(x, y, t) \mapsto (x, y, \alpha/t)$ , for some  $\alpha$  satisfying  $\alpha^3 = \Delta(F)/\Delta(E)$ . Taking a third involution  $(x, y, t) \mapsto (x, y, \alpha\zeta_3/t)$  yields a rank 16 subgroup of  $MW(\pi_6)$ , isomorphic to  $MW(\pi_6)/MW(\pi_1)$ . (This is discussed in more detail in the Subsections 4.3 and 4.6.) In Subsection 4.6 we also describe the field of definition of sections generating  $MW(\pi_6)$  modulo  $MW(\pi_1)$ . It turns out that our description corrects a mistake in Kuwata's description of the minimal field of definition.

**RESULTS 2.4.** On the Kuwata surfaces  $X_1, X_2, \dots, X_6$  one has several automorphisms such that the obtained quotients are rational surfaces. This is used in Section 4 to give the following results:

- We give sufficient conditions on  $E, F$  and  $K$  to have a rank 1 or a rank 2 subgroup of  $MW_K(\pi)$ .
- We give an indication how one can find explicit generators of  $MW(\pi_2)/MW(\pi_1)$ . The precise generators can be found in [32].
- We give an indication how one obtains a degree 24 polynomial such that its zeroes determine a set of generators of  $MW(\pi_3)/MW(\pi_1)$ . The explicit polynomial can be found in [32].
- We give explicit generators of a subgroup of finite index in  $MW(\pi_4)/MW(\pi_2)$ . (See Corollary 4.12.)
- We give an indication how one obtains a degree 240 polynomial such that its zeroes determine a set of generators of  $MW(\pi_5)/MW(\pi_1)$ . We did not manage to write down this polynomial. The degree of this polynomial and the number of variables is too high to write it down explicitly. For several choices of  $a, b, c, d$  it can be found in [32].
- We give an algorithm for determining a set of generators of  $MW(\pi_6)/MW(\pi_3)$ . The explicit generators are given in [32].

If  $E$  and  $F$  are not isogenous then  $MW(\pi_1) = 0$  (see Theorem 2.1). If  $E$  and  $F$  are isogenous then the degree of the  $x$ -coordinate of a generator of  $MW(\pi_1)$  seems to depend on the degree of the isogeny between  $E$  and  $F$ . This seems to give an obstruction for obtaining explicit formulas for all cases.

Before providing the explicit equations, we mention the following result, which can be proven without knowing explicitly the generators of  $MW(\pi_i)$ .

**PROPOSITION 2.5.** *Suppose that  $E$  and  $F$  are defined over  $\mathbf{Q}$ . One has that*

$$\text{rank } MW_{\mathbf{Q}}(\pi_i) \leq \begin{cases} 1 & \text{if } i = 1, \\ 5 & \text{if } i = 2, \\ 7 & \text{if } i = 3, \\ 9 & \text{if } i = 4, \\ 5 & \text{if } i = 5, \\ 11 & \text{if } i = 6. \end{cases}$$

The total contribution of  $\pi_1, \pi_2, \pi_3, \pi_4, \pi_6$  to  $MW_{\mathbf{Q}}(\pi_{12})$  is bounded by 15. The maximum total contribution of  $\pi_1, \dots, \pi_6$  to  $MW_{\mathbf{Q}}(\pi_{60})$  is bounded by 19.

PROOF. If  $\text{rank } MW_{\mathbf{Q}}(\pi_1)$  equals 2 then the Shioda-Tate formula 1.2.11 implies that  $NS(X)$  has rank 20 and  $NS(X)$  is generated by divisors defined over  $\mathbf{Q}$ . It is well-known that this is impossible ([71]).

Suppose that  $i \geq 2$ . Let  $S$  be the image of a section of  $\pi_i$ , such that  $S$  is not the strict transform of the pull back of a section of  $\pi_j$ , for some  $j$  dividing  $i$ , and  $j \neq i$ . It is easy to check that if we push forward  $S$  to  $X_1$  and then pull this divisor back to  $X_i$ , we obtain a divisor  $D$  consisting of  $i$  geometrically irreducible components, and one of these components is  $S$ .

Set  $M_j := MW_{\mathbf{Q}}(\pi_j)$ . Set  $M := M_i / \sum_{j>0, j|i, j \neq i} M_j$ . Take a minimal set of sections  $S_m, m = 1, \dots, \ell$ , satisfying the following property: denote  $S_{m,n}$  the components of the pull back of the push forward of  $S_m$ , then the  $S_{m,n}$ , with  $m = 1, \dots, \ell$  and  $n = 1, \dots, i$ , generate  $M \otimes \mathbf{Q}$ . Consider the  $\ell n$ -dimensional  $\mathbf{Q}$ -vector space  $F$  of formal linear expression in the  $S_{m,n}$ . By definition of the  $S_n$  it follows that the natural map  $\Psi : F \rightarrow M \otimes \mathbf{Q}$  is surjective. Fix a generator  $\sigma$  of the (birational) automorphism group of the rational map  $X_i \rightarrow X_1$ . We can split  $F \otimes \mathbf{C}$  and  $M_j \otimes \mathbf{C}$  into eigenspaces for the action of  $\sigma$ . Set  $E_{\zeta_i^k} \subset F \otimes \mathbf{C}$  the eigenspace for the eigenvalue  $\zeta_i^k$ . Set  $V' := \bigoplus_{j|\gcd(i,j)=1} E_{\zeta_i^j}$ . Since the  $S_{i,j}$  form a minimal set, we have that that  $\Psi|_{V'}$  is injective. It is easy to see that if  $\gcd(i, k) \neq 1$  then the eigenspace  $E_{\zeta_i^k}$  is contained in the kernel of  $\Psi$ , hence we obtain that  $\Psi|_{V'}$  is an isomorphism. This implies that  $\dim V' = \varphi(i)\ell$ , where  $\varphi$  is the Euler  $\varphi$ -function.

Fix  $m$  such that  $1 \leq m \leq \ell$ . Let  $G$  be the subgroup of  $MW(\pi_i)$  generated by the  $S_{m,n}$  for  $n = 1, \dots, i$ . There is a faithful action of the Galois group  $\text{Gal}(\mathbf{Q}(\zeta_i)/\mathbf{Q})$  on  $G$ . This implies that the sections defined over  $\mathbf{Q}$  form a rank at most 1 subgroup of  $G$  (if  $i$  is odd) or a rank at most two subgroup of  $G$  (if  $i$  is even). From this one obtains that

$$M_{\mathbf{Q}} := MW_{\mathbf{Q}}(\pi_i) / \bigoplus_{j>0, j|i, j \neq i} MW_{\mathbf{Q}}(\pi_j)$$

has rank at most  $\ell$  (when  $i$  is odd) or  $2\ell$  (when  $i$  is even). This combined with Proposition 2.1 provides the upper bounds for  $MW_{\mathbf{Q}}(\pi_i)$  for  $i = 2, 3, 5, 6$ . The upper bound for  $MW_{\mathbf{Q}}(\pi_4)$  follows from the fact that the above mentioned Galois action is non-trivial, hence for each  $i$  there is an element  $g$  in  $G$  not fixed under the action of  $\text{Gal}(\mathbf{Q}(\zeta_i)/\mathbf{Q})$  and  $g$  is not mapped to 0 in  $M$ . One easily obtains from this that  $M_{\mathbf{Q}}$  has co-rank at least  $\ell$  in  $M$ , which gives the case  $i = 4$ .

The final assertions follow from a reasoning as in the proof of Corollary 2.2.  $\square$

For some of the  $\pi_i$  the bounds in Proposition 2.5 are sharp, for some of the others we do not know:

- We have  $\text{rank } MW_{\mathbf{Q}}(\pi_1) = 1$  when  $E$  and  $F$  are isogenous over  $\mathbf{Q}$ . (see 4.1.)
- We have that  $\text{rank } MW_{\mathbf{Q}}(\pi_2) \geq 4$  if and only if  $E$  and  $F$  have complete two-torsion over  $\mathbf{Q}$ , and, moreover, if  $E$  and  $F$  are isogenous then  $\text{rank } MW_{\mathbf{Q}}(\pi_2) = 5$ . (see 4.2.)
- There exists  $E$  and  $F$  such that  $\text{rank } MW_{\mathbf{Q}}(\pi_4) \geq 8$ . (see 4.4.)
- There exists  $E$  and  $F$  such that  $\text{rank } MW_{\mathbf{Q}}(\pi_6) \geq 6$ .
- If one can find a rational solution of  $uv^2(u^4 - 1)(v^4 - 1)^2 = w^3$ , satisfying  $-w(u - v) \neq 0$  and

– if  $p_1 = 4u^2/(u^2 - 1)$  and  $p_2 = 4v^2/(v^2 - 1)$  then  $p_1 \notin \{p_2, 1/p_2, 1 - p_2, 1 - 1/p_2, p_2/(1 - p_2), 1/(1 - p_2)\}$ .

then one can make examples with  $\text{rank } MW_{\mathbf{Q}}(\pi_{12}) \geq 10$ . (see 4.6.)

REMARK 2.6. The highest known rank over  $\mathbf{Q}$  for an elliptic surface  $\pi : X \rightarrow \mathbf{P}^1$  over  $\mathbf{Q}$  is 14 [30]. This example is a member of a family of elliptic surface introduced by Mestre [44].

The actual aim of this chapter is to produce explicit equations for the generators of  $MW(\pi_i)$  modulo  $MW(\pi_1)$ , for  $i = 2, \dots, 6$ . For convenience, we always assume that  $E$  and  $F$  have complete 2-torsion over the base field. It is not so hard to deduce from our results, these equations for the case that  $E$  and  $F$  have 2-torsion points defined over a larger field.

The organization of this chapter is as follows. In Section 3 we indicate an algorithm for finding generators of  $MW(\pi)$  in the case that of a rational elliptic surface  $\pi : X \rightarrow \mathbf{P}^1$  with an additive fiber. In Section 4 we prove the results stated in 2.4. To find generators for the  $MW(\pi_i)$  we often rely on the results of Section 3.

### 3. Finding sections on a rational elliptic surface with an additive fiber

Fix a field  $K$  of characteristic 0 and a rational elliptic surface  $\pi : X \rightarrow \mathbf{P}^1$  over  $K$ . Such a surface can be represented by a Weierstrass equation

$$(3) \quad y^2 = x^3 + (\alpha_4 t^4 + \alpha_3 t^3 + \alpha_2 t^2 + \alpha_1 t + \alpha_0)x + \beta_6 t^6 + \beta_5 t^5 + \beta_4 t^4 + \beta_3 t^3 + \beta_2 t^2 + \beta_1 t + \beta_0,$$

with  $\alpha_i, \beta_j \in K$ . Assume that over  $t = \infty$  the fiber is singular and of additive type. This happens if and only if  $\alpha_4 = \beta_6 = 0$ .

From [53, Theorem 2.5] we know that  $MW(\pi)$  is generated by sections of the form

$$(4) \quad x = b_2 t^2 + b_1 t + b_0, y = c_3 t^3 + c_2 t^2 + c_1 t + c_0.$$

Substituting (4) in (3) and using  $\alpha_4 = \beta_6 = 0$  yields that

$$c_3^2 = b_2^3.$$

Set  $c_3 = p_1^3, b_2 = p_1^2$ . First we search for solutions with  $p_1 \neq 0$ . The equation for the coefficient of  $t^5$  is an equation of the form  $p_1^2 c_2 + f$  with  $f$  a polynomial in  $p_1, b_1, \alpha_i, \beta_j$ . Hence we can express  $c_2$  in terms of the  $p_1, b_1, \alpha_i, \beta_j$ . Similarly, we can express  $c_1$  and  $c_0$  in terms of  $p_1, b_1, \alpha_i, \beta_j$ . One easily shows that if this procedure fails, then  $p_1 = 0$ .

Unfortunately, the three remaining equations  $F_m = 0$  in  $b_1, b_0, p_1, \alpha_i, \beta_j$  are not linear in any of the variables, but the degree of the  $F_i$  in  $b_1, b_0, p_1$  is sufficiently low to compute the resultant

$$R(p_1) := \text{res}_{b_1}(\text{res}_{b_0}(F_1, F_3), \text{res}_{b_0}(F_2, F_3))$$

in concrete examples (i.e., after substituting constants for the  $\alpha_i, \beta_j$ ). Calculating  $R(p_1)$  and finding zeroes of it gives all possibilities for  $p_1$ . Substituting such a value of  $p_1$  in  $\text{res}(F_1, F_3, b_0)$  and in  $\text{res}(F_2, F_3, b_0)$  yields two polynomials in  $b_1$ . Calculating the g.c.d. of these polynomials gives a list of possible values for  $b_1$ . Then substituting all possibilities for  $(p_1, b_0)$  in  $F_1, F_2, F_3$  gives all possibilities for  $p_1, b_0, b_1$ .

Consider the case  $p_1 = 0$ , hence  $c_3 = b_2 = 0$ . The coefficient of  $t^5$  is zero if and only if  $\alpha_5 = 0$ . If this is the case then the coefficient of  $t^4$  is of the form  $-c_2^2 + \beta_3 b_1 + \alpha_4$ .

If  $\beta_3 \neq 0$ , then one eliminates  $b_1, b_0, c_0$  as is done above, yielding two polynomial equation in two unknowns. In this case it suffices to compute only one resultant.

If  $\beta_3 = 0$  then we can substitute  $c_2 = \pm\sqrt{\alpha_4}$ . The coefficient of  $t^3$  is of the form  $-2\sqrt{\alpha_4}c_1 + f$ , with  $f$  a polynomial in  $c_0, \alpha_i, \beta_j, \sqrt{\alpha_4}$ . We can solve  $c_1$  and obtain  $-2\sqrt{\alpha_4}c_0 + g$  as a coefficient for  $t^3$ , where  $g$  is a polynomial in  $c_0, \alpha_i, \beta_j, \sqrt{\alpha_4}$ .

This fails when  $\alpha_4 = 0$ . In that case  $c_2 = 0$ , and one has four polynomial relations in  $b_0, c_0, c_1, b_1$ . We can eliminate  $c_1$  as above, yielding three polynomial relations in three unknowns which can be solved as above. In this way we find all possible sections of the form (4).

Top [80, Section 5] discusses the following elliptic surface:

**THEOREM 3.1.** *Let  $\pi : X \rightarrow \mathbf{P}^1$  be the elliptic surface*

$$y^2 = x^3 + 108(27t^4 - 74t^3 + 84t^2 - 48t + 12).$$

*Then  $MW_{\mathbf{Q}}(\pi)$  has rank 3 and is generated by sections  $\sigma_i$  with  $x$ -coordinates*

$$x(\sigma_1) = 6t, \quad x(\sigma_2) = 6t - 8, \quad x(\sigma_3) = -12t + 9.$$

*and fix a primitive cube root of unity  $\omega$ . Let  $\tau_i$  be obtained from  $\sigma_i$  by multiplying the  $x$ -coordinate with  $\omega$ . Then the  $\sigma_i$  and  $\tau_i$  generate a subgroup of finite index of  $MW(\pi)$ .*

Top found the sections  $\pm\sigma_1, \pm\sigma_2, \pm\tau_1$  and  $\pm\tau_2$ ; an explicit description of the third independent section  $\sigma_3$  seems not to be present in the literature.

**PROOF.** The elliptic surface  $\pi : X \rightarrow \mathbf{P}^1$  has 4 fibers of type *II* and at  $t = \infty$  a fiber of type *IV*. From this it follows that  $MW(\pi)$  has rank 6. Top [80, Section 5] observes that the Mordell-Weil group is generated by a subset of the 18 sections of the form  $(\omega x_i, \pm y_i)$ ,  $i = 1, 2, 3$ , with  $\omega^3 = 1$  and  $x_i$  is a polynomial of degree 1.

The form of the generators imply that, in terms of the above discussion, we are looking for sections with  $p_1 = 0$ . After eliminating all the  $c_i$  and  $b_j$ , except for  $b_1$ , we obtain a polynomial of degree 27. It can be factored as the product of  $(b_1 - 6)^2(b_1 + 12)$  times 12 polynomials of degree 2, where all the degree 2 factors have discriminant  $-3$ .

This implies that  $x = 6t$ ,  $x = 6t - 8$  and  $x = -12t + 9$  are the only  $x$ -coordinates of degree 1 defined over  $\mathbf{Q}$ . These sections are disjoint from the zero-section, and intersect at  $t = \infty$  the singular fiber in the non-identity component. One can choose the  $y$ -coordinates of the three sections in such a way that the third sections is disjoint from the first two. This implies that the height pairing (see [70, Definition 8.5]) yields the following intersection matrix

$$\begin{pmatrix} 4/3 & 2/3 & 2/3 \\ 2/3 & 4/3 & 2/3 \\ 2/3 & 2/3 & 4/3 \end{pmatrix}.$$

The determinant of this matrix is non-zero. This implies that these three sections generate a rank 3 subgroup  $G_1$ , and  $G_1 = MW_{\mathbf{Q}}(\pi)$ .

Multiplying the  $x$ -coordinates with a cube root of unity will yield another rank 3 subgroup  $G_2$  corresponding to a different eigenspace of the action of complex multiplication hence  $G_1 \oplus G_2$  generate a rank 6 subgroup of the Mordell-Weil group.  $\square$

#### 4. Explicit formulas

Fix a field  $K$  of characteristic 0. Let  $E : y^2 = x^3 + ax + b$  and  $F : y^2 = x^3 + cx + d$  be elliptic curves, with complete two-torsion over  $K$ . Assume that  $E$  and  $F$  have distinct

$j$ -invariant. Fix  $\lambda, \mu, \nu$  and  $\xi$  such that  $E$  is isomorphic to  $y^2 = x(x - \lambda)(x - \mu)$  and  $F$  is isomorphic to  $y^2 = x(x - \nu)(x - \xi)$ . Then we may assume that

$$a = \frac{1}{3}(\lambda\mu - \lambda^2 - \mu^2), \quad b = \frac{1}{27}(3\lambda\mu(\lambda + \mu) - 2(\lambda^3 + \mu^3)),$$

and similar equations for  $c$  and  $d$ .

The Kummer surface  $Y$  is birational to the surface  $S \subset \mathbf{A}^3$  given by  $(x^3 + ax + b)t^2 = z^3 + cz + d$ , and the fibration  $\psi : Y \rightarrow \mathbf{P}^1$  corresponds to the map  $(x, z, t) \mapsto t$ .

LEMMA 4.1. *The groups  $MW(\pi_i), i \geq 1$  are torsion-free.*

PROOF. Kuwata [41, Theorem 4.1] shows that  $\pi_6$  has smooth fibers over  $t = 0, \infty$  and only singular fibers of type  $I_1$  or  $II$ . Hence the same holds for  $\pi_{6i}$ . This fact together with [47, Corollary VII.3.1] implies that the group  $MW(\pi_{6i})$  is torsion-free. Since  $MW(\pi_i)$  is a subgroup of  $MW(\pi_{6i})$  it is also torsion-free.  $\square$

We now discuss how to find explicit formulas for generators of  $MW(\pi_i), i = 1, \dots, 6$ .

4.1.  $\pi_1 : X_1 \rightarrow \mathbf{P}^1$ . It is not easy to find explicit equations for sections on  $\pi_1$ , since there are infinitely many cases, depending on the minimal degree of an isogeny  $E \rightarrow F$ . Instead we give a sufficient condition to have rank 1 or 2 over  $K$ .

LEMMA 4.2. *The group  $MW_K(\pi_1)$  has rank at most 2.*

- If  $E$  and  $F$  are isogenous over  $K$  then  $MW_K(\pi_1)$  has positive rank.
- If  $E$  and  $F$  are isogenous over  $K$  and  $E$  admit complex multiplication then  $MW_K(\pi_1)$  has rank 2.
- If  $MW_K(\pi_1)$  has positive rank then there exists a degree at most two extension  $L/K$  such that  $E$  and  $F$  are isogenous.
- If  $MW_K(\pi_1)$  has rank 2 then there exists a degree at most two extension  $L'/L$  such that  $E$  admits complex multiplication over  $L'$ .

PROOF. Since  $E$  and  $F$  have complete two torsion it follows that  $\text{rank } MW_K(\pi_1) = r$  if and only if  $18 + r = \text{rank } NS_K(X_2) = \text{rank } NS_K(Y)$ , with  $Y$  the Kummer surface of  $E$  and  $F$ .

It is easy to see that if  $E$  and  $F$  satisfy the first, resp., second assumption then the rank of  $NS_K(Y)$  is at least 19, resp., 20.

If  $NS_K(Y) \geq 19$  then  $E$  and  $F$  are isogenous over some extension of  $K$ . Let  $\Gamma$  be the graph of the isogeny. Then the push-forward of  $\Gamma$  on  $Y$  is Galois-invariant. This implies that  $([-1] \times [-1])^* \Gamma + \Gamma$  is Galois-invariant. From this it follows that  $E$  and  $F$  are isogenous over a degree 2 extension.

If  $NS_K(Y) = 20$  then  $E$  and  $F$  are isogenous over some extension of  $K$  and  $E$  has potential complex multiplication. Let  $\Gamma$  be the graph of an isogeny, let  $\Gamma'$  be the graph of the isogeny composed with complex multiplication. Then the push-forward of both  $\Gamma$  and  $\Gamma'$  are Galois-invariant. As above, this implies that  $E$  admits complex multiplication over a degree 2 extension of  $L$ .  $\square$

REMARK 4.3. Suppose  $E$  and  $F$  are elliptic curves, not isogenous over  $K$ , but isogenous over a degree 2 extension  $L/K$ . Suppose that  $\text{End}(E) = \mathbf{Z}$ . Let  $\varphi : E \rightarrow F$  be an isogeny defined over  $L$ . It is an easy exercise to show that the divisor

$$D := \{(P, \varphi(P)) \mid P \in E\} \cup \{(P, -\varphi(P)) \mid P \in E\} \subset E \times F$$



is invariant under the action of  $\text{Gal}(L/K)$ . Hence the push-forward of  $D$  onto the Kummer is invariant under the action of  $\text{Gal}(\bar{K}/K)$ . The argument used in the above proof gives that  $MW_K(\pi_1)$  has rank at least 1, while  $E$  and  $F$  are not isogenous over  $K$ .

**4.2.**  $\pi_2 : X_2 \rightarrow \mathbf{P}^1$ . Since  $MW(\pi_2)$  is torsion-free, we have that  $MW(\pi'_1) \oplus MW(\pi_1)$  is of finite index in  $MW(\pi_2)$ , where  $\pi'_1 : X'_1 \rightarrow \mathbf{P}^1$  is the twist of  $\pi_1$  at 0 and  $\infty$ . (For more on this topic see Section 1.5.)

One can easily show that  $MW(\pi'_1) \subset MW(\pi'_3)$ , where  $\pi'_3$  is the twist of  $\pi_3$  at 0 and  $\infty$ . Since  $MW(\pi'_3)$  can be considered in a natural way as a subgroup of  $MW(\pi_6)$ , we refer to that subsection for a discussion of the results. The explicit equations for generators of  $MW(\pi'_2)$  are given in [32]. If we drop for a moment the condition that  $E$  and  $F$  have complete 2-torsion over  $K$ , then we can use the result in [32] to prove that  $MW_K(\pi'_1)$  has rank 4 if and only if  $E$  and  $F$  have complete two-torsion over  $K$ .

**4.3.**  $\pi_3 : X_3 \rightarrow \mathbf{P}^1$ . A Weierstrass equation for  $\pi_3$  is

$$y^2 = x^3 - 48acx + (\Delta(F)t^3 + 864bd + \Delta(E)t^{-3}).$$

Setting  $s = (t + \alpha_i/t)$ , with  $\alpha_i^3 = \Delta(E)/\Delta(F)$  and  $i = 1, 2, 3$ , will give an equation for the rational elliptic surface  $\psi_i : S_i \rightarrow \mathbf{P}^1$ , given by

$$y^2 = x^3 - 48acx - \Delta(F)(s^3 - 3\alpha_i s) + 864bd.$$

This surface has a fiber of type  $I_0^*$  at  $s = \infty$ . For each choice of  $\alpha_i$  we obtain an isomorphic surface over  $K(\zeta_3)$ , but the pullback of the sections to  $\pi_3$  depends on the choice of  $\alpha_i$ :

LEMMA 4.4. *For  $i \neq j$  we have  $MW(\psi_i) \cap MW(\psi_j) = \{\sigma_0\}$ , where  $MW(\psi_i)$  and  $MW(\psi_j)$  are considered as subgroups of  $MW(\pi_3)$ .*

PROOF. Without loss of generality, we may assume that  $i = 1$  and  $j = 2$ . Set  $L := K(\zeta_3, \alpha_1)$ . Let  $G \subset \text{Aut}(L(t))$  be generated by  $\varphi_1 : t \mapsto \alpha_1/t$  and  $\varphi_2 : t \mapsto \alpha_1\zeta_3/t$ . From  $\varphi_1 \circ \varphi_2 : t \mapsto t\zeta_3$  it follows that  $\#G \geq 6$ . Set  $s' = t^3 + \alpha_1^3/t^3$ . Then  $s'$  is fixed under  $G$ . We have the following inequalities

$$6 \leq \#G = [L(t)^G : L(t)] \leq [L(s') : L(t)] = 6.$$

These inequalities give that  $L(t)^G = L(s')$ . This implies that a section in  $MW(\psi_i) \cap MW(\psi_j)$  is the pull back of a section of the elliptic surface  $\psi'$  with Weierstrass equation

$$y^2 = x^3 - 48acx - \Delta(F)s' + 864bd.$$

This is an equation of rational elliptic surface with a  $II^*$ -fiber. In this case the Shioda-Tate formula 1.2.11 implies that  $MW(\psi')$  has rank 0. Since  $MW(\psi')$  is a subgroup of the torsion-free group  $MW(\pi_3)$  (see Lemma 4.1) it follows that  $\#MW(\psi') = 1$ .  $\square$

LEMMA 4.5. *We have that  $\text{rank } MW(\psi_i) = 4$ .*

PROOF. From the equation of  $\psi_i$  one easily sees that it is a rational elliptic surface, with a fiber of type  $I_0^*$  over  $s = \infty$ , and no other reducible singular fibers. Hence the Shioda-Tate formula 1.2.11 implies that  $\text{rank } MW(\psi_i) = 4$ .  $\square$

LEMMA 4.6. *For  $i \neq j$  we have  $(MW(\psi_i) \oplus MW(\psi_j)) \cap MW(\pi_1) = \{\sigma_0\}$ , considered as subgroups of  $MW(\pi_3)$ .*

PROOF. From Lemma 4.4 and Lemma 4.5 it follows that  $MW(\psi_i) \oplus MW(\psi_j)$  injects into  $MW(\pi_3)$ . Consider the vector spaces  $V = (MW(\psi_i) \oplus MW(\psi_j)) \otimes \mathbf{C}$  and  $W = MW(\pi_1) \otimes \mathbf{C}$ . The automorphism  $\sigma : (x, y, t) \mapsto (x, y, \zeta_3 t)$  induces a trivial action on  $W$ . We now prove that  $\sigma$  maps  $V$  to itself and all eigenvalues of this action are different from 1.

Assume for the moment that  $\lambda, \mu, \nu$  and  $\xi$  are algebraically independent over  $\mathbf{Q}$ . This defines a Kuwata elliptic surface

$$\Pi_k : \mathcal{X}_k \rightarrow \mathbf{P}^1_{\mathbf{Q}(\lambda, \mu, \nu, \xi, \sqrt[3]{\lambda(\lambda-\mu)\mu/\nu(\nu-\xi)\xi})}$$

and, similarly,

$$\Psi_k : \mathcal{S}_k \rightarrow \mathbf{P}^1_{\mathbf{Q}(\lambda, \mu, \nu, \xi, \sqrt[3]{\lambda(\lambda-\mu)\mu/\nu(\nu-\xi)\xi})}.$$

Since  $MW(\Pi_1) = 0$ , and  $MW(\Psi_i) \oplus MW(\Psi_j)$  has rank 8, it follows that a lift  $\tilde{\sigma}$  of  $\sigma$  acts on  $\tilde{V} := MW(\Psi_i) \oplus MW(\Psi_j)$ , hence  $\sigma$  acts on  $V$ . If  $\tilde{\sigma}$  would have an eigenvalue 1 on  $\tilde{V}$ , then  $MW(\Pi_1)$  would be non-trivial. Hence also  $\sigma$  acts without eigenvalue 1. This implies that  $V \cap W = 0$ , yielding the lemma.  $\square$

From the Lemmas 4.4, 4.5 and 4.6 and Theorem 2.1 it follows that for  $i \neq j$  we have that  $MW(\psi_i) \oplus MW(\psi_j)$  generates a subgroup of finite index in  $MW(\pi_3)/MW(\pi_1)$ . Hence if one wants to describe the Galois representation on  $MW(\pi_3)/MW(\pi_1)$  it suffices to describe the Galois representation on  $MW(\psi_i)$ .

Since  $\psi_i : S_i \rightarrow \mathbf{P}^1$  are rational elliptic surfaces with an additive fiber at  $s = \infty$  we can apply Section 3 to calculate expressions for the generators. The relevant formulae for this may be found in [32].

**4.4.**  $\pi_4 : X_4 \rightarrow \mathbf{P}^1$ . Let  $\pi'_2 : X'_2 \rightarrow \mathbf{P}^1$  be the twist of  $\pi_2$  by the points 0 and  $\infty$ . (Compare Section 1.5.) The results of Section 1.5 imply that the group  $MW(\pi_2) \oplus MW(\pi'_2)$  is of finite index in  $MW(\pi_4)$ . Since the fibers over 0 and  $\infty$  of  $\pi_2$  are of type  $IV^*$  and all other fibers are of type  $I_1$  or  $II$ , it follows from the results mentioned in Section 1.5 that  $\pi'_2$  has only fibers of type  $II$  or  $I_1$ . Moreover,  $\pi'_2 : X'_2 \rightarrow \mathbf{P}^1$  defines a rational elliptic surface, hence  $\text{rank } MW(\pi'_2) = 8$ .

LEMMA 4.7. *Let  $\pi : X \rightarrow \mathbf{P}^1$  be a Jacobian elliptic surface such that all singular fibers are irreducible. Let  $Z := \sigma_0(\mathbf{P}^1)$ . Let  $S$  be the image of a section  $\sigma : \mathbf{P}^1 \rightarrow X$ . Denote  $S_n$  the image of  $(n\sigma) : \mathbf{P}^1 \rightarrow X$ .*

*Then the equality  $(S_n \cdot Z) = (S_m \cdot Z)$  holds if and only if  $n = \pm m$ .*

PROOF. Let  $\langle \cdot, \cdot \rangle$  denote the height pairing on  $MW(\pi)$  (see [70, Definition 8.5]). In this case  $\langle T, T \rangle = 2(T \cdot Z) - 2\chi(X)$ , hence

$$2(S_n \cdot Z) - 2\chi(X) = \langle S_n, S_n \rangle = \langle nS_1, nS_1 \rangle = n^2 \langle S_1, S_1 \rangle = 2n^2(S_1 \cdot Z) - 2n^2\chi(X).$$

It follows that  $(S_n \cdot Z) = n^2((S \cdot Z) - \chi(X)) + \chi(X)$ , which yields the lemma.  $\square$

Assume, for the moment, that  $\lambda, \mu, \nu$  and  $\xi$  are algebraically independent over  $\mathbf{Q}(t)$ . We can consider the elliptic surface  $\pi'_2$  as an elliptic curve  $A$  over  $K' := \mathbf{Q}(\lambda, \mu, \nu, \xi, t)$ .

Then on  $A$  we have that

$$x = \frac{2(\sqrt{\lambda} + \sqrt{\mu})\lambda\mu}{\sqrt{\xi} + \sqrt{\xi - \nu}} \frac{1}{t} + \left( -\frac{4}{3}(2\xi - \nu)(\lambda + \mu) + 4\sqrt{\lambda\mu\xi(\xi - \nu)} \right) + \frac{2(\sqrt{\xi} + \sqrt{\xi - \nu})\xi(\xi - \nu)}{\sqrt{\lambda} + \sqrt{\mu}} t$$

is a  $x$ -coordinate of a point  $P_1$ , hence giving rise to 2 different points on  $A$  (see [32]). If we plug this  $x$ -coordinate in the equation for  $A$ , then

$$y^2 = 2(\sqrt{\xi} + \sqrt{\xi - \nu})(\sqrt{\lambda} + \sqrt{\mu})h(t)^2$$

for some  $h \in \mathbf{Q}(t, \sqrt{\lambda}, \sqrt{\mu}, \sqrt{\xi}, \sqrt{\nu - \xi})$  (see [32]).

Using  $P_1$  we now show how to find points  $P_2, \dots, P_8$ , such that if we specialize, the  $P_i$  generate  $MW(\pi'_2)$ . Let  $P_2$  be a section such that  $x(P_2)$  is obtained from  $x(P_1)$  by replacing  $\sqrt{\mu}$  by  $-\sqrt{\mu}$ . (The existence follows from the fact that the map  $\sqrt{\mu} \mapsto -\sqrt{\mu}$  fixes the equation of  $\pi'_2$ .)

LEMMA 4.8. *The points  $P_1$  and  $P_2$  are independent in  $A(\overline{K})$ .*

PROOF. Since  $x(P_1) \neq x(P_2)$  it follows that  $P_1 \neq \pm P_2$ . Hence the sections  $S_1$  and  $S_2$  on  $\pi'_2$  satisfy  $(S_1 \cdot Z) = (S_2 \cdot Z)$  and  $S_1 \neq \pm S_2$ . It follows from Lemma 4.7 that  $S_1$  and  $S_2$  generate a subgroup of  $MW(\pi'_2)$  of rank 2, hence  $P_1$  and  $P_2$  are independent.  $\square$

Note that we have  $P_1 \in A(K_+)$  and  $P_2 \in A(K_-)$ , where

$$K_{\pm} := K'(\sqrt{\lambda}, \sqrt{\mu}, \sqrt{\xi}, \sqrt{\xi - \nu}, \sqrt{2(\sqrt{\xi} + \sqrt{\xi - \nu})(\sqrt{\mu} \pm \sqrt{\lambda})}).$$

The automorphism  $\sigma$  of  $K'$ , fixing  $\mathbf{Q}$  and further given by

$$(t, \lambda, \mu, \nu, \xi) \mapsto \left( \frac{1}{t}, \nu, \xi, \lambda, \mu \right)$$

is an automorphism of  $A$ , (i.e., swapping  $E$  and  $F$ , and applying the  $\mathbf{P}^1$ -automorphism  $t \mapsto 1/t$ ). This yields to different points  $P_3 := \sigma(P_1), P_4 := \sigma(P_2)$  on  $A$ .

LEMMA 4.9. *The points  $P_1, \dots, P_4$  generate a rank 4 subgroup of  $A(\overline{K})$ .*

PROOF. The points  $P_3$  and  $P_4$  are not in  $A(K_+K_-)$ , but are contained in  $A(K'_{\pm})$ , where

$$K'_{\pm} = K'(\sqrt{\mu}, \sqrt{\mu - \lambda}, \sqrt{\nu}, \sqrt{\xi}, \sqrt{2(\sqrt{\mu} + \sqrt{\mu - \lambda})(\sqrt{\xi} \pm \sqrt{\nu})}).$$

Applying [63, Lemma 1.3.2] yields that the  $P_i$  generate a rank 4 subgroup.  $\square$

Let  $\tau'$  automorphism of  $K'$  fixing  $\mathbf{Q}(t)$  and mapping  $(\xi, \lambda) \leftrightarrow (\nu, \mu)$  (i.e., we are interchanging the role of the two-torsion points of both  $E$  and  $F$ ). Set  $P'_i = \tau(P_i)$ .

LEMMA 4.10. *The points  $P_1, \dots, P_4, P'_1, \dots, P'_4$  generate a subgroup of rank 8 of  $A(\overline{K})$ .*

PROOF. The points  $P'_1, \dots, P'_4$  are defined over different fields (i.e., they are defined over  $K_{\pm}(i)$  or  $K'_{\pm}(i)$ , but not over  $K_{\pm}$  or  $K'_{\pm}$ .) From [63, Lemma 1.3.2] it follows then that the rank of the subgroup generated by the  $S_i, S'_i$  is 8.  $\square$

PROPOSITION 4.11. *The sections of  $\pi'_2$  associated to the points  $P_1, \dots, P_4, P'_1, \dots, P'_4$  of  $A$  generate a rank 8 subgroup of  $MW(\pi'_2)$ .*

PROOF. Let  $\Pi'_2 : \mathcal{X}'_2 \rightarrow \mathbf{P}^1_{\mathbf{Q}(\lambda, \mu, \nu, \xi)}$  be the elliptic surface corresponding to  $A/K'$ . The points  $P_i, P'_i$  define sections  $\mathcal{S}_i, \mathcal{S}'_i$  of  $\Pi'_2$ , which generate a rank 8 subgroup of  $MW(\Pi'_2)$ . This implies that for a general specialization the specialized sections  $S_i, S'_i$  generate a rank 8 subgroup of  $MW(\pi'_2)$ . Since the intersection numbers are constant under deformation, it follows that if the  $S_i, S'_i$  do not generate  $MW(\pi'_2)$ , then  $\pi'_2$  has at least one reducible fiber. From [41, Theorem 4.3] it follows that then  $E$  and  $F$  are isomorphic, which contradicts our assumption at the beginning of this section.  $\square$

COROLLARY 4.12. *The sections of  $\pi'_2$  associated to the points  $P_1, \dots, P_4, P'_1, \dots, P'_4$  generate a subgroup of finite index in  $MW(\pi_4)/MW(\pi_2)$ .*

PROOF. This follows from the above Proposition together with the observation that the direct sum of  $MW(\pi'_2)$  and  $MW(\pi_2)$  generates a subgroup of finite index of  $MW(\pi_4)$ .  $\square$

We will now take special values for  $\lambda, \mu, \nu, \xi$  such that several sections are defined over  $\mathbf{Q}$ .

COROLLARY 4.13. *For infinitely many pairs of elliptic curves  $(E, F)$  over  $\mathbf{Q}$ , the group  $MW_{\mathbf{Q}}(\pi_4)$  has rank 8 or 9.*

PROOF. We have that  $\text{rank } MW_{\mathbf{Q}}(\pi_4) = \text{rank } MW_{\mathbf{Q}}(\pi_2) + \text{rank } MW_{\mathbf{Q}}(\pi'_2)$ . From the results in 4.1 and 4.2 we know that  $\text{rank } MW_{\mathbf{Q}}(\pi_2)$  is either 4 or 5.

As mentioned in Section 2, one can prove that the rank over  $\mathbf{Q}$  of  $\pi'_2$  is at most 4. To obtain rank 4 it suffices to choose  $\lambda, \mu, \nu, \xi$  such that  $S_1, S_2, S_3$  and  $S_4$  are defined over  $\mathbf{Q}$ . In order to obtain this set  $\lambda = l^2, \mu = m^2, \nu = n^2, \xi = k^2, k^2 - n^2 = n_2^2$  and  $m^2 - l^2 = l_2^2$ . Then

$$2(k + n_2)(m \pm l), 2(m + l_2)(k \pm n)$$

have to be a non-zero square for all choices of  $\pm$ . One easily computes that this occurs precisely when

$$l_2 n_2 = \frac{u^2(\rho - 1)(\tau^2 - 1)}{4(\rho + 1)\tau^2}$$

for some  $u \in \mathbf{Q}^*$  and

$$k = n_2 \frac{\tau^2 + 1}{\tau^2 - 1}, \quad n = n_2 \frac{2\tau}{\tau^2 - 1}, \quad m = l_2 \frac{\rho^2 + 1}{\rho^2 - 1}, \quad l = l_2 \frac{2\rho}{\rho^2 - 1}.$$

From these last equations we can obtain our original  $\lambda, \mu, \nu, \xi$ .

The associated Legendre parameters are

$$\left( \frac{2\tau}{\tau^2 + 1} \right)^2 \quad \text{and} \quad \left( \frac{2\rho}{\rho^2 + 1} \right)^2.$$

One easily finds  $\rho, \tau$  such that the corresponding  $j$ -invariants are different.  $\square$

For later use, we remark that  $\Delta(E)/\Delta(F)$  is a third power if and only if

$$\frac{\tau(\tau^4 - 1)}{\rho(\rho^4 - 1)}$$

is a third power. A necessary condition for obtaining curves with different  $j$ -invariant is  $\rho \neq \tau$ . The only solution we found with these properties is  $(\rho, \tau) \in \{(2, 3), (3, 2)\}$ . This gives rise to two curves having both Legendre parameter  $25/9$ , hence does not give an interesting solution.

**4.5.**  $\pi_5 : X_5 \rightarrow \mathbf{P}^1$ . An equation for  $\pi_5$  is

$$y^2 = x^3 - 48acx + (\Delta(F)t^5 + 864bd + \Delta(E)t^{-5})$$

Setting  $s = (t + \alpha_i/t)$ , with  $\alpha_i^5 = \Delta(E)/\Delta(F)$ , gives a rational elliptic surface  $\psi_i : S_i \rightarrow \mathbf{P}^1$ .

We can now copy the strategy used for  $\pi_3$ . We have that  $MW(\pi_5)/MW(\pi_1)$  has rank 16, the  $MW(\psi_i)$  have rank 8, the intersection  $MW(\psi_i) \cap MW(\psi_j) = \{\sigma_0\}$ , considered as subgroups of  $MW(\pi_5)$  and  $(MW(\psi_i) \oplus MW(\psi_j)) \cap MW(\pi_1) = \{\sigma_0\}$ . This combined with the fact that all the  $\psi_i$  are isomorphic over  $K(\zeta_5)$  implies that it suffices to find sections of only one of the  $\psi_i$ . One can show that  $\psi_i$  is a rational elliptic surface with an additive fiber at  $t = \infty$ . Hence we can apply Section 3 to find an expression for the sections of  $MW(\pi_5)$ . In all choices for  $\lambda, \mu, \nu, \xi$  we tried the final resultant is a product of two polynomials of degree 120. (See [32])

**4.6.**  $\pi_6 : X_6 \rightarrow \mathbf{P}^1$ . Since  $\pi_6$  is torsion-free we have that  $MW(\pi_3) \oplus MW(\pi'_3)$  is of finite index in  $MW(\pi_6)$ , where  $\pi'_3 : X'_3 \rightarrow \mathbf{P}^1$  is the twist of  $\pi_3$  at 0 and  $\infty$ .

The group  $MW(\pi_3)$  is described above. We discuss here how to find generators for  $MW(\pi'_3)$ . We follow Kuwata [41, Section 5]. Kuwata observes that  $X'_3$  is birational over  $\mathbf{Q}$  to the cubic surface in  $\mathbf{P}^3$

$$C : Z^3 + cZY^2 + dY^3 = X^3 + aXW + bW^3,$$

and that the strict transforms of the 27 lines of  $C$  to  $X'_3$  generate  $MW(\pi'_3)$ .

It has been known for a long time how to find the 27 lines on a cubic surface of the above form, see for example [61]. Due to our special situation we give a somewhat different approach to find all 27 lines.

In our case  $C$  is isomorphic to

$$(*) \quad Z(Z - \nu Y)(Z - \xi Y) = X(X - \lambda W)(X - \mu W).$$

One finds 9 lines defined over  $\mathbf{Q}$ , namely the intersections of  $Z - \alpha Y = 0$  and  $X - \beta W = 0$ , with  $\alpha \in \{0, \nu, \xi\}$  and  $\beta \in \{0, \lambda, \mu\}$ . We give now the equations for the strict transforms of these lines on  $X_3$ :

LEMMA 4.14. *Set*

$$A = (a_{ki}) = \begin{pmatrix} \nu - \xi & \xi - \nu & \nu \\ -\xi & -\nu & \xi \\ -\mu & -\lambda & \lambda \\ \lambda - \mu & \mu - \lambda & \mu \end{pmatrix}.$$

Then

$$x = 4a_{1i}a_{2i}t^2 + \frac{4}{3}(a_{1i} + a_{2i} + a_{3j} + a_{4j})t + 4a_{3j}a_{4j}$$

and

$$y = 4a_{1i}a_{2i}(a_{1i} + a_{2i})t^3 + 8a_{1i}a_{2i}(a_{3j} + a_{4j})t^2 + 8a_{3j}a_{4j}(a_{1i} + a_{2i})t + 4a_{3j}a_{4j}(a_{3j} + a_{4j})$$

for  $1 \leq i, j \leq 3$  are sections of  $\pi'_3$ .

PROOF. This is a straightforward computation. See [32].  $\square$

The usual strategy to find the other 18 lines is to take pencils of planes through the lines we found above. We choose a different strategy. The image of a section on  $X'_3$  can be pulled back to a divisor on  $X_6$ . We can take the push-forward to  $Y$  of this divisor and then push-forward in a natural way to  $\mathbf{P}^1 \times \mathbf{P}^1$ . We give 6 divisors on  $\mathbf{P}^1 \times \mathbf{P}^1$ , such that

the pull back of such a divisor to  $X_6$  consists of three components, and the 18 divisors obtained in this way correspond to the 18 lines on  $C$ . Consider  $\mathbf{P}^1 \times \mathbf{P}^1$  with projective coordinates  $X, W$  and  $Y, Z$ . Set

$$P_1 := [0, 1], P_2 := [\lambda, 1], P_3 := [\mu, 1], Q_1 := [0, 1], Q_2 := [\nu, 1], Q_3 := [\xi, 1].$$

Fix a permutation  $\sigma \in S_3$ . Let  $C_\sigma \subset \mathbf{P}^1 \times \mathbf{P}^1$  be the  $(1, 1)$ -curve going through  $(P_i, Q_{\sigma(i)})$ . For example,  $C_{\text{id}}$  is given by

$$G(X, Y, Z, W) := \nu\xi(\lambda - \mu)XW + \mu\lambda(\xi - \nu)WZ + (\nu\mu - \xi\lambda)XZ = 0.$$

We describe what the corresponding divisor on  $C$  is. This can be done by dehomogenizing first, say, we set  $W = Y = 1$ . Then we can express  $Z$  in terms of  $X$ . Substitute this expression for  $Z$  in  $(*)$ . We obtain a rational expression in  $X$  (a quotient of a polynomial of degree 6 and a polynomial of degree 3). The numerator is zero if and only if  $X = 0, X = \lambda, X = \mu$  or  $X$  satisfies a degree 3 polynomial  $f$ . For example if  $\sigma = \text{id}$  then

$$\left( \frac{\mu\lambda(\nu - \xi) + \sqrt[3]{\mu\lambda(\mu - \lambda)\nu^2\xi^2(\nu - \xi)^2}}{\nu\mu - \xi\lambda} \right) \zeta_3^k =: \gamma_k$$

are the zeroes of  $f$ . Let  $H$  be the hyperplane  $X = \gamma_k Y$ . Let  $H'$  be  $G(X, Y, \gamma_k, 1) = 0$ . Then  $H \cap H'$  is contained in  $C$  and defines a line  $\ell_{\sigma, k}$ . One can easily show that for all  $\sigma \in H$  one obtains three such hyperplanes, and that one obtains 18 lines in total.

REMARK 4.15. One can show that 6 of these 18 lines are defined over

$$K(E[2], F[2], \sqrt[3]{\Delta(E)/\Delta(F)}),$$

and the 12 others over

$$K(E[2], F[2], \mu_3, \sqrt[3]{\Delta(E)/\Delta(F)}).$$

Moreover, one shows easily that if one of the 18 lines is defined over  $K(E[2], F[2], \mu_3)$ , then  $\Delta(E)/\Delta(F) \in K(E[2], F[2])^{*3}$ . This contradicts the claim in [41, Section 5], which states that all 27 lines are defined over  $K(E[2], F[2], \mu_3)$ .

REMARK 4.16. One can easily find elliptic curves  $E, F$  over  $\mathbf{Q}$  with complete two-torsion defined over  $\mathbf{Q}$  such that  $\Delta(E)/\Delta(F) \in \mathbf{Q}^{*3}$ : one needs to find solutions of

$$\lambda(\lambda - \mu)\mu = \tau^3\nu(\nu - \xi)\xi.$$

This defines a conic over  $\mathbf{Q}(\tau, \mu, \xi)$  containing the rational point  $\lambda = 0$  and  $\nu = 0$ . Hence one can parameterize this conic and find solution giving rise to elliptic curves. For example  $(\lambda, \mu, \nu, \xi, \tau) = (16, 1, 6, 1, 2)$  gives an example.

REMARK 4.17. Suppose one can find a solution in  $\mathbf{Q}^3$  of

$$\frac{\tau(\tau^4 - 1)}{\rho(\rho^4 - 1)} = \sigma^3$$

satisfying the conditions mentioned at the end of 4.4. Then we obtain examples such that  $\text{rank } MW_{\mathbf{Q}}(\pi_6) \geq 6$  and  $\text{rank } MW_{\mathbf{Q}}(\pi_4) \geq 8$ , yielding  $\text{rank } MW_{\mathbf{Q}}(\pi_{12}) \geq 10$ .

